Details of TPM – Discreteness Correlation

1.) Discreteness Correlation Function

$$\tilde{f}(1) = \frac{1}{n} \sum_{i=1}^{N} \delta(x_1 - x_i(t))\, \delta(v_1 - v_i(t))$$

$$\langle\ \rangle = \int dx_i \int dv_i\ n \langle f \rangle$$

$$\begin{cases} 1 = x_1, v_1, t \\ 2 = x_2, v_2, t \end{cases}$$

$$\langle\ \rangle = \int dx_i \int dv_i\, n\langle f \rangle$$

$$\langle \tilde{f}(1) \tilde{f}(2) \rangle =$$

$$= \int dx_i \int dv_i\ n \left(\frac{1}{n} \sum_{i=1}^{N} \delta(\underline{x}_1 - \underline{x}_i(t))\, \delta(\underline{v}_1 - \underline{v}_i(t)) \right) \left(\frac{1}{n} * \sum_{j=1}^{N} \delta(\underline{v}_2 - \underline{v}_j(t))\, \delta(\underline{x}_2 - \underline{x}_j(t)) \right)$$

$$= \int dx_i \int dv_i\ \frac{\langle f \rangle}{n} \sum_{i,j=1}^{N} \left[\delta(\underline{x}_1 - \underline{x}_i(t))\, \delta(\underline{x}_2 - \underline{x}_j(t)) * \delta(\underline{v}_1 - \underline{v}_i(t))\, \delta(\underline{v}_2 - \underline{v}_j(t)) \right]$$

only $\neq 0$ if arguments interchangeable

$$= \int dx_i \int dv_i\ \frac{\langle f \rangle}{n} \sum_{i,j=1}^{N} \left[\delta(\underline{x}_1 - \underline{x}_2)\, \delta(\underline{x}_i - \underline{x}_j)\, \delta(\underline{v}_1 - \underline{v}_2) * \delta(\underline{v}_i - \underline{v}_j) \right]$$

$$= \frac{\langle f \rangle}{n}\, \delta(\underline{x}_1 - \underline{x}_2)\, \delta(\underline{v}_1 - \underline{v}_2)$$

Now seek:

$$C(k,\omega) = \int dv_1 \int dv_2 \, \langle \tilde{f}(1)\,\tilde{f}(2) \rangle_{k,\omega}$$

$$= \int dv_1 \int dv_2 \int d\underline{x}_- \, e^{-ikx_-} \int d\tau \, e^{i\omega\tau} \langle \tilde{f}(0)\,\tilde{f}(x,\tau) \rangle$$

→ where have used: $\left\{ \begin{array}{l} \text{homogeneity} \\ \text{stationarity} \end{array} \right.$

→ $v_\pm = (\underline{v}_2 \pm \underline{v}_1)/2$

so

$$\int d\underline{v} \langle \tilde{f}(1)\,\tilde{f}(2) \rangle_{k,\omega} = \frac{\langle f \rangle}{n} \int e^{i\omega\tau} e^{-ik\cdot x(\tau)} \, d\tau$$

$\underline{x}(\tau) = \underline{v}\tau$ → ballistic, u.p.o.

⇒
$$\boxed{\int d\underline{v} \langle \tilde{f}^2 \rangle_{k,\omega} = 2\pi\,\delta(\omega - \underline{k}\cdot\underline{v}) \, \frac{\langle f \rangle}{n}}$$

k,ω orbit propagator

↳ weight function ⟶ particle distribution

so
$$\boxed{C(k,\omega) = \int dv\, \frac{\langle f \rangle}{n}\, 2\pi\, \delta(\omega - k\cdot v)}$$
→ noise spectrum

so, can write in 1D: (hereafter)

$$C(k,\omega) = \frac{2\pi}{n\,|k|\,v_{Te}} \left\langle \bar{f}(\omega/k v_{Te}) \right\rangle$$

↳ v_{Te} factor extracted

⇓

discreteness noise has Maxwellian Doppler spectrum...

and

$$\boxed{\langle \hat{\phi}^2 \rangle_{k,\omega} = \left(\frac{4\pi n_0 |e|}{k^2}\right)^2 \frac{1}{n_0}\, \frac{2\pi}{|k|\,v_{Te}}\, \frac{\left\langle \bar{f}(\omega/k v_{Te}) \right\rangle}{|\epsilon(k,\omega)|^2}}$$

∴ have obtained thermal equ'b'm spectrum.

Can re-write as:

Maxwellian Dist. Scatter

$$\langle \phi^2 \rangle_{k,\omega} = \underbrace{n_0}_{\#\text{ particles}} \underbrace{\left(\frac{4\pi|e|}{k^2}\right)^2}_{\text{Coulomb Spectrum}} \frac{2\pi}{|k|v_{Te}} \frac{\langle \bar{f}(\omega/kv_{Te}) \rangle}{\underbrace{|\epsilon(k,\omega)|^2}_{\text{screening}}}$$

→ Spectrum defined / determined by:

- eqbrm. particle emission spectrum $\sim e^{-\omega^2/k^2 v_{Te}^2}$
- collective response
$$\epsilon \cong 1 - \frac{\omega_p^2}{\omega^2} + i\epsilon_{IM} \quad (\omega > kv_{Te})$$
$$\epsilon \cong 1 + 1/k^2\lambda_D^2 + i\epsilon_{IM} \quad (\omega < kv_{Te})$$

→ Can note:

1) collective response strongest at wave resonance $(kV \sim \omega \sim \omega_{pe})$

⇒ expect peak at natural frequency.

2) $\omega \gg \omega_p$ $(v \gg \omega_p/k)$ $\Rightarrow$ noise source decouples from collective dynamics (i.e. plasma as vacuum)

$$\langle \hat{\phi}^2 \rangle_{k,\omega} \cong n_0 \left(\frac{4\pi |e|}{k^2} \right)^2 \frac{2\pi}{|k| v_{Te}} e^{-\omega^2/k^2 v_{Te}^2}$$

3) $\omega \ll \omega_p$ $\Rightarrow$ static limit (Debye screening)

$$\langle \hat{\phi}^2 \rangle_{k,\omega} \cong n_0 (4\pi |e|)^2 \frac{2\pi}{|k| v_{Te}} e^{-\omega^2/k^2 v_{Te}^2} \Big/ \left(k^2 + 1/\lambda_D^2\right)^2$$

So, can write Electric Field Spectrum:

$$\boxed{\frac{|\hat{E}_{k,\omega}|^2}{8\pi} = \frac{4\pi^2 n_0 |e|^2}{k^2 |k|} \left\{ \frac{F(\omega/k v_{Te})}{\underbrace{\left(1 - \frac{\omega_p^2}{\omega^2}\right)^2}_{|\epsilon_r|^2} + \underbrace{\left(\frac{\pi \omega_{pe}^2}{|k| k} F'\right)^2}_{|\epsilon_{Im}|^2}} \right\}}$$

$|\epsilon_r|^2 + |\epsilon_{Im}|^2$

$F' \sim$ slope.

i.e. $F' = dF/d\left(\frac{\omega}{k}\right)$

Now, to make contact with usual expectations:

i.e. "$k_B T/2$ per-degree-of-freedom"

$$W_k \equiv \int d\omega \, |\hat{E}_{k,\omega}|^2 / 8\pi$$

field energy per mode

Useful trick: Pole Approximation

$$\frac{1}{|\epsilon|^2} \cong \frac{1}{(\omega-\omega_k)^2 \left|\frac{\partial\epsilon}{\partial\omega}\right|^2 + |\epsilon_{IM}|^2}$$

$|\epsilon_{IM}|^2$ → sets width

$(\omega-\omega_k)$ → real freq. sets location

$$\cong \frac{1}{|\epsilon_{IM}|} \left\{ \frac{|\epsilon_{IM}|}{(\omega-\omega_k)^2 \left|\frac{\partial\epsilon}{\partial\omega}\right|^2 + |\epsilon_{IM}|^2} \right\}$$

$$\cong \frac{1}{|\epsilon_{IM}|} \left| (\partial\epsilon_r/\partial\omega)_{\omega_k} \right|^{-1} \delta(\omega-\omega_k)$$

$$\boxed{\frac{1}{|\epsilon|^2} \cong \delta(\omega-\omega_k) \Big/ |\epsilon_{IM}(k,\omega_k)| \left|\frac{\partial\epsilon_r}{\partial\omega}\right|_{\omega_k}}$$

so, integrating in pole approximation:

$$W_k = \frac{m_e \omega_{pe}}{2|k|} \frac{F}{|F'|}$$

$$\cong \frac{m_e \omega_{pe}}{2|k|} \frac{\cancel{F}}{\frac{\omega_{pe}}{|k| v_{Te}^2} \cancel{F}} = \frac{T}{2} \checkmark$$

in accord with "T/2 per degree" intuition.

→ if $k\lambda_D \gg 1$ $(\epsilon \to 1 \text{ limit})$

$$W_k \cong \frac{T}{2} 1/k^2\lambda_D^2$$ → strong cut-off beyond λ_D.

so, for total energy density:

$$\left\langle \frac{E^2}{8\pi} \right\rangle = \int dk\, W_k = \int \frac{d\underline{k}}{(2\pi)^3} \frac{T/2}{1+k^2\lambda_D^2}$$

$$\sim \left(\frac{k_b T}{2}\right) k_{max}^3$$

$$\sim \left(\frac{n k_b T}{2}\right) \frac{1}{n\lambda_D^3}$$

so $\frac{\langle E^2 \rangle}{8\pi} \sim \left(\frac{n k_B T}{2} \right) \frac{1}{n \lambda_D^3}$

$\sim$ (Particle Kinetic Energy Density) $/ n \lambda_D^3$

\# particles in Debye sphere

$\Rightarrow$

(FED) $\sim \frac{\text{(PKED)}}{n \lambda_D^3}$

$1/n\lambda_D^3$ ~ discreteness factor

→ To connect formally, to fluctuation-dissipation theorem:

note: $$\epsilon_{IM} = -\frac{\omega_p^2 \pi}{k|k|} \left.\frac{\partial \langle f \rangle}{\partial v}\right|_{\omega/k}$$

$$= \frac{2\pi\omega}{k^2 v_{Te}^2} \frac{\omega_p^2}{|k| v_{Te}} \langle \bar{f}(\omega/k) \rangle \quad , \quad \text{for Maxwellian } \langle f \rangle$$

so $$\langle \bar{f}(\omega/k) \rangle = k^2 v_{Te}^2 |k| v_{Te} \, \epsilon_{IM} \Big/ 2\pi\omega\omega_p^2$$

and have:

$$\langle \hat{\phi}^2 \rangle_{k,\omega} = \frac{2\pi n}{|k| v_{Te}} \left(\frac{4\pi |e|}{k^2}\right)^2 \frac{\langle \bar{f}(\omega/k) \rangle}{|\epsilon(k,\omega)|^2}$$

so, plugging in:

$$\langle \hat{\phi}^2 \rangle_{k,\omega} = \frac{8\pi T}{k^2 \omega} \frac{\text{Im}\,\epsilon}{|\epsilon|^2}$$

and $$\boxed{\frac{\langle \hat{E}^2 \rangle_{k,\omega}}{8\pi} = \frac{T}{\omega} \frac{\text{Im}\,\epsilon}{|\epsilon|^2}} \Leftrightarrow \left\{ \begin{array}{l} \text{Fluctuation-} \\ \text{Dissipation} \\ \text{Theorem} \end{array} \right.$$

(restates form of spectrum)

→ relates thermal fluctuations to <u>dissipation</u> in collective modes ($\text{Im}\,\epsilon$)

→ obviously consistent (by construction) with physical picture of thermal equilibrium b.) Some general comments:

→ Key element of T.P.M. is use of linear $f^c_{k,\omega}$ or, equivalently, unperturbed orbits.

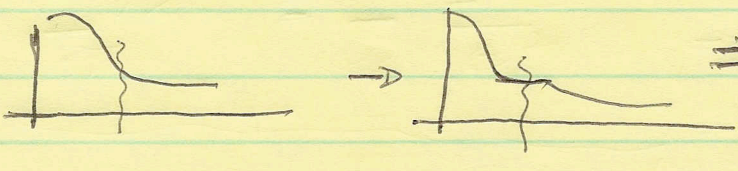
- This assumes small fluctuation levels, so stochastic deflection is 'weak'

i.e. $\underline{x}(t) = \underline{x}(0) + \underline{v}t + \cancel{\delta(\underline{x}(t))}$
deflection

How weak? ⟺ take care $\tau_{ac} < \tau_T$ condition

<u>but</u> - $\langle \hat{\phi}^2 \rangle_k \sim (\;) \dfrac{F(\omega_{pe}/k)}{|F'(\omega_{pe}/k)|}$

∴ fluctuations diverge as $F' \to 0$, from below

i.e.  ⟹ "Critical Opalescence"

Note $F' > 0$ <u>not</u> necessary ⟺ theory fails for <u>stable</u> <u>plasma</u>....., approaching marginality.

→ As fluctuations grow, linearizations fail

∴ must renormalize:

→ particle propagator

$$i/\omega - kv \rightarrow i/\omega - kv + \underbrace{\Sigma}_{\text{self energy, decorrelation rate}}$$

→ mode propagator / response

$$1/\epsilon \rightarrow 1\Big/\left[\underbrace{(\omega - (\omega_k + \delta\omega_k))}_{\text{nonlinear frequency shift}}\frac{\partial \epsilon}{\partial \omega} + i\underbrace{(\epsilon_{Im}^{L} + \epsilon_{Im}^{NL})}_{\text{nonlinear dissipation}}\right]$$

(w-w interaction
w-p interaction)

⇒ γ^{NL}

(recall NL oscillator, driven!)

Calculating all this is aim of plasma turbulence theory.....